Calculus 3 / Multivariable Calculus[1], Part 1 of 2

Towards and through the vector fields

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A short table of contents

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An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

[2] Chapter numbers in Robert A. Adams, Christopher Essex: *Calculus, a complete course.* 8th or 9th edition.

21 Completing the square. Problems 4 and 5.

Problem 4: Identify the curve defined by the following equation: $x^2 - 2x - y^2 - 2y = 1$ (a hyperbola).

Problem 5: Identify the curve defined by the following equation: $4x^2 - 4y^2 + 8y - 5 = 0$ (a hyperbola).

Extra material: notes with solved problems 4 and 5.

22 Completing the square. Problems 6 and 7.

Problem 6: Identify the curve defined by the following equation: $x^2 - 2x - y = -2$ (a parabola).

Problem 7: Identify the curve defined by the following equation: $y^2 - 2y - x = 0$ (a parabola).

Extra material: notes with solved problems 6 and 7.

S4 Quadric surfaces (spheres, cylinders, cones, ellipsoids, paraboloids etc) (Ch.10.5)

23 Quadric surfaces, an introduction.

24 Quadric surfaces, degenerate quadrics.

Extra material: notes (cylinder, cone).

25 Quadric surfaces, ellipsoids.

26 Quadric surfaces, paraboloids.

Extra material: notes (paraboloids and their cross-sections).

27 Quadric surfaces, hyperboloids.

Analyze and draw the hyperboloid of one sheet $-x^2 + y^2 + z^2 = 4$.

Analyze and draw the hyperboloid of two sheets $x^2 - y^2 - z^2 = 4$.

Extra material: notes (hyperboloids and their cross-sections).

28 Problem solving, quadric surfaces. Problems 1 and 2.

Problem 1: Identify the surface defined by the following equation: $x^2 + y^2 + z^2 = 2z$ (a sphere).

Problem 2: Identify the surface defined by the following equation: $x^2 + 4y^2 + 9z^2 = 36$ (an ellipsoid).

29 Problem solving, quadric surfaces. Problem 3.

Problem 3: Identify the surface defined by the following equation: $x^2 + y^2 - 6x + 4y - z + 10 = 0$ (a paraboloid).

30 Problem solving, quadric surfaces. Problems 4 and 5.

Problem 4: Identify the surface defined by the following equation: $y^2 = 4x^2 + 16z^2$ (an elliptic cone).

Problem 5: Identify the surface defined by the following equation: $x = z^2 + z$ (a parabolic cylinder).

31 Problem solving, quadric surfaces. Problem 6.

Problem 6: Identify the surface defined by the following equation: $x^2 + y^2 - z^2 - 4x - 6y + 2z + 12 = 0$ (an elliptic cone).

Extra material: notes with solved problems 1–6.

S5 Topology in $\mathbb{R}^n$: distance, open ball, neighborhood, open and closed set, inner and outer point, boundary point. (Ch.10.1)

32 Some topology: neighborhoods.

33 Some topology: open, closed, bounded sets.

34 How to identify sets in the plane from their mathematical description, an introduction.

35 How to identify sets in the plane from their mathematical description; Example 1.

Problem 1: Draw the set described by: $D = \{(x, y);\ \ -x^2+2x < y < 8-x^2\}$. Is the set open/closed/neither? Is the set bounded?

36 How to identify sets in the plane from their mathematical description; Example 2.

Problem 2: Draw the set described by: $D = \{(x, y);\ \ x^2 \leqslant y \leqslant \sqrt{x}\}$. Is the set open/closed/neither? Is the set bounded?

37 How to identify sets in the plane from their mathematical description; Example 3.

Problem 3: Draw the set described by: $D = \left\{ (x, y) \mid x > 0,\ \ 0 \leqslant y < 2,\ \ y \leqslant \frac{1}{x} \right\}$. Is the set open/closed/-neither? Is the set bounded?

38 How to identify sets in the plane from their mathematical description; Example 4.

Problem 4: Draw the set described by: $D = \{\ (x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1 + x^4, \quad 0 \leqslant y + x\ \}$. Is the set open/closed/neither? Is the set bounded?

39 How to identify sets in the plane from their mathematical description; Example 5.

Problem 5: Draw the set described by: $D = \{\ (x, y);\ \ 1 \leqslant x^2 + y^2 \leqslant 4,\ \ 0 \leqslant x \leqslant y \leqslant \sqrt{3}x\ \}$. Is the set open/closed/neither? Is the set bounded?

40 How to identify sets in the plane from their mathematical description; Examples 6 and 7.

Problem 6: Draw the set described by: $D = \{(x, y) \mid\ -1 \leqslant x \leqslant 1,\ \ x^2 - 4 \leqslant y,\ \ y \leqslant e^x,\ \ y \leqslant e^{-x}\}$. Is the set open/closed/neither? Is the set bounded?

Problem 7: Draw the set described by: $D = \{(x, y) \mid\ \ 1 \leqslant x^2 + y^2, \quad -e^{-x^2} \leqslant y \leqslant e^{-x^2}\}$. Is the set open/closed/neither? Is the set bounded?

S6 Coordinates: Cartesian, polar, cylindrical, spherical coordinates (Ch.10.6)

41 Different coordinate systems.

42 Other coordinate systems: polar coordinates in the plane.

Equation $r = c$ for any positive constant c describes a circle with centre in the origin.

Equation $\theta = \theta_0$ for any $\theta_0 \in [0,\ 2\pi)$ describes a half-line without the origin.

43 Polar coordinates in the plane: an important example.

44 Problem solving: polar coordinates.

Problem 1: Convert the Cartesian coordinates $(2, -2)$ to polar coordinates.

Problem 2: Identify the curve defined by the equation $r = \frac{1}{1 - \cos\theta}$ (a parabola).

Problem 3: Describe the set D in polar coordinates: $D = \{\ (x, y);\ \ 1 \leqslant x^2 + y^2 \leqslant 4,\ \ 0 \leqslant x \leqslant y \leqslant \sqrt{3}x\ \}$.

Extra material: notes with solved problems 1, 2 and 3.

45 Other coordinate systems: cylindrical coordinates in the space.

46 Problem solving: cylindrical coordinates. Pr.1.

Problem 1: Convert the Cartesian coordinates $(2, -2, 1)$ to cylindrical coordinates.

47 Problem solving: cylindrical coordinates. Pr.2.

Problem 2: Convert the equation $zr = 2 - r^2$ written in cylindrical coordinates into an equation in Cartesian coordinates.

48 Problem solving: cylindrical coordinates. Pr.3.

Problem 3: Identify the surface defined by the equation: $z = 3 - r^2$ (a paraboloid).

49 Problem solving: cylindrical coordinates. Pr.4.

Problem 4: Identify the surface defined by the equation: $r^2 - 4r\cos\theta = 5$ (a cylinder).

Extra material: notes with solved problems 1–4.

50 Other coordinate systems: spherical coordinates in the space.

51 Other coordinate systems: spherical coordinates in the space, some examples.

Equation $R = c$ for any positive constant c describes a sphere with centre in the origin.

Equation $\theta = \theta_0$ for any $\theta_0 \in [0,\ 2\pi)$ describes a vertical half-plane with edge along the z-axis, without this edge.

Equation $\phi = \phi_0$ describes a cone, the xy-plane or a half-line, depending on ϕ_0.

52 Other coordinate systems: spherical coordinates in the space; conversion.

53 Problem solving: spherical coordinates. Pr.1.

Problem 1: Convert the Cartesian coordinates $(2, -2, 1)$ to spherical coordinates.

54 Problem solving: spherical coordinates. Pr.2.

Problem 2: Convert the cylindrical coordinates $[2,\ \pi/6,\ -2]$ to Cartesian and to spherical coordinates.

55 Problem solving: spherical coordinates. Pr.3.

Problem 3: Convert the spherical coordinates $[4,\ \pi/3,\ 2\pi/3]$ to Cartesian and to cylindrical coordinates.

56 Problem solving: spherical coordinates. Pr.4.

Problem 4: Describe the "ice cream cone" in spherical coordinates: $x^2 + y^2 \leqslant z^2,\ x^2 + y^2 + z^2 \leqslant 1,\ z \geqslant 0$.

Extra material: notes with solved problems 1–4.

You will learn: to understand which geometrical objects are represented by simpler equations and inequalities in $\mathbb{R}^2$ and $\mathbb{R}^3$, determine whether a set is open or closed, if a point is an inner, outer or boundary point, determine the boundary points, describe points and other geometrical objects in the different coordinate systems.

C1 Vector-valued functions, parametric curves

(Chapter 11: 11.1, 11.3)

S7 Introduction to vector-valued functions

57 Curves: an introduction.

58 Functions: repetition.

59 Vector-valued functions, parametric curves.

60 Vector-valued functions, parametric curves: domain.

What is the domain of $\vec{r}(t) = \left(2t+1,\ \dfrac{\sin t}{t},\ \dfrac{1}{\sqrt{1-t^2}}\right)$?

S8 Some examples of parametrization

61 Vector-valued functions, parametric curves: parametrization.

parametrization of circles

parametrization of ellipses

parametrization of graphs to $y = f(x)$.

parametrization of graphs to $x = g(y)$.

62 Parametric curves: an intriguing example.

Identify the curve given by the following parametrization: $\vec{r}(t) = (\cos 2t,\ \sin t),\quad t \in [0,\ 2\pi]$.

Extra material: notes with the solution of the problem above.

63 Parametric curves: problem solving 1.

Identify the curve with following parametrization: $x = \dfrac{1}{1+t^2},\quad y = \dfrac{t}{1+t^2},\quad t \in \mathbb{R}$.

64 Parametric curves: problem solving 2.

Identify the curve with following parametrization: $\mathbf{r}(t) = \left(\dfrac{1-t^2}{1+t^2},\ \dfrac{2t}{1+t^2}\right),\quad t \in \mathbb{R}$.

Extra material: notes with the solution of the problem above.

65 Parametric curves: problem solving 3.

Find a parametrization of the intersection between the paraboloid $z = x^2 + y^2$ and the plane $x + z = 2$.

Find a parametrization of the intersection between the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$.

66 Helix.

Draw the curve $\mathbf{r}(t) = (\cos t,\ \sin t,\ t),\quad t \in [0,\ 2\pi]$.

Extra material: notes with the solution of the problem above.

S9 Vector-valued calculus; curve: continuous, differentiable and smooth

67 Vector-valued calculus: notation.

68 Vector-valued calculus: limit and continuity.

69 Vector-valued calculus: derivatives.

Compute the derivative of $\vec{r}(t) = (2t+1,\ \ \sin^2 t,\ \ e^t \cos t)$.

70 Vector-valued calculus: speed, acceleration.

71 Position, velocity, acceleration: an example.

Compute velocity and acceleration of a particle moving along the path described by $\vec{r}(t) = (t^2,\ \sin t),\ t \geqslant 0$.

72 Vector-valued calculus: smooth and piecewise smooth curves.

Is the following curve smooth?: $\vec{r}(t) = (t^3,\ t^2)\ \ t \in \mathbb{R}$.

73 Vector-valued calculus: sketching a curve.

Sketch the following curve: $\vec{r}(t) = (t^2-4,\ t^3-4t),\quad -2.5 \leqslant t \leqslant 2.5$.

74 Another curve to sketch: an exercise.

Sketch the following curve: $\vec{r}(t) = \left(t^2,\ \ \frac{1}{3}t(t^2-3)\right),\quad -2 \leqslant t \leqslant 2$.

75 Problem solving 2 vector-valued calculus. Pr.1.

Problem 1: Position, speed, velocity, acceleration for $\vec{r}(t) = (t^2,\ \ -t^2,\ \ 1)$.

76 Problem solving 2 vector-valued calculus. Pr.2.

Problem 2: Position, speed, velocity, acceleration for $\vec{r}(t) = (3\cos\omega t,\ \ 4\cos\omega t,\ \ 5\sin\omega t)$.

77 Problem solving 2 vector-valued calculus. Pr.3.

Problem 3: A particle moves around the circle $x^2+y^2=25$ at constant speed, making one revolution in 2s. Find its acceleration when it is at $(3,4)$.

Extra material: notes with solved problems 1–3.

78 Vector-valued calculus: extra theory 1 (limit and continuity).

79 Vector-valued calculus: extra theory 2 (derivative, tangent, velocity).

80 Vector-valued calculus: differentiation rules.

Extra material: proofs of some of the differentiation rules.

81 Vector-valued calculus: differentiation rules. Example 1.

Example 1: Expand and simplify: $\dfrac{d}{dt}\left((\vec{u}+\vec{u}'') \cdot (\vec{u}\times\vec{u}')\right)$.

Extra material: notes with the solution of the problem above.

82 Vector-valued calculus: differentiation rules. Example 2.

Example 2: A point P moves along the curve of intersection of the cylinder $z=x^2$ and the plane $x+y=2$ in the direction of increasing y with constant speed $v=3$. Find the velocity of P when it is at $(1,1,1)$.

Extra material: notes with the solution of the problem above.

83 Position, velocity, acceleration. Example 3.

Example 3: The speed is constant on some interval if and only if the acceleration is perpendicular to the velocity on this interval.

Extra material: notes with the solution of the problem above.

84 Position and velocity. One more example.

Example 4: If the position and velocity of a moving particle are always perpendicular, show that the path of the particle lies on a sphere.

Extra material: notes with the solution of the problem above.

85 Position, velocity, acceleration: trajectories of planets

Example 5: If the motion is caused by a force pointing towards the same point (the origin) then the whole path of the motion lies in some plane.

Extra material: notes with the solution of the problem above.

S10 Arc length

86 Parametric curves: arc length.

87 Arc length: problem solving. Problem 1.

Problem 1: Compute the arc length of the curve $\vec{r}(t) = \left(t^2, \ \frac{1}{3}t(t^2-3) \right), \ \ t \in [0, \ \sqrt{3}]$.

88 Arc length: problem solving. Problems 2 and 3.

Problem 2: Compute the arc length of the curve $\vec{r}(t) = (t^2, \ t^2, \ t^3), \ \ t \in [0,1]$.

Problem 3: Compute the arc length of the curve $\vec{r}(t) = (t^3, \ t^2), \ \ t \in [0,1]$.

89 Arc length: problem solving. Problems 4 and 5.

Problem 4: Compute the arc length of the curve $\vec{r}(t) = (e^t, \ t\sqrt{2}, \ e^{-t}), \ \ t \in [0,3]$.

Problem 5: Compute the arc length of the curve $\vec{r}(t) = (2\cos t, \ \ 2\sin t, \ \ \frac{2}{3}t^{3/2}), \quad t \in [0, 2\pi]$.

Extra material: notes with the solutions of the problems above.

S11 Arc length parametrization

90 Parametric curves: parametrization by arc length.

91 Parametric curves: parametrization by arc length, how to do it. Example 1.

Problem 1: Find the arc length parametrization of the helix defined by $\mathbf{r}(t) = (\cos t, \ \sin t, \ t)$.

Extra material: notes with the solution of the problem above.

92 Parametric curves: parametrization by arc length. Example 2.

Problem 2: Find the arc length parametrization for $\mathbf{r}(t) = (\ln(1+t^2), \ \ln(1+t^{-2}), \ \sqrt{8}\arctan t), \ \ t \in [1, T]$.

Extra material: notes with the solution of the problem above.

93 Arc length does not depend on parametrization. Theory.

Extra material: notes with the proof.

You will learn: Parameterize some curves (straight lines, circles, ellipses, graphs of functions of one variable); if $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a function describing a particle's position in $\mathbb{R}^3$ with respect to time t, describe position, velocity, speed and acceleration; compute arc length of parametric curves; arc length parametrization.

C2 Functions of several variables; differentiability

(Chapter 12)

S12 Real-valued functions in multiple variables, domain, range, graph surface, level curves, level surfaces

You will learn: describe the domain and range of a function, Illustrate a function $f(x, y)$ with a surface graph or with level curves.

94 Functions of several variables, introduction.

95 Functions of several variables, introduction, continuation 1.

96 Functions of several variables, introduction, continuation 2.

97 Functions of several variables, domain.

98 Functions of several variables, domain. Problem solving 1.

Problem 1: Specify the domain of the function $f(x,y) = \frac{x+y}{x-y}$.

Problem 2: Specify the domain of the function $f(x,y) = \sqrt{xy}$.

Problem 3: Specify the domain of the function $f(x,y) = \frac{x}{x^2+y^2}$.

Problem 4: Specify the domain of the function $f(x,y) = \frac{xy}{x^2-y^2}$.

Extra material: notes with the solutions of the problems above.

99 Functions of several variables, domain. Problem solving 2.

Problem 5: Specify the domain of the function $f(x,y) = \sqrt{4x^2+9y^2-36}$.

Problem 6: Specify the domain of the function $f(x,y) = \frac{1}{\sqrt{4x^2+9y^2-36}}$.

Extra material: notes with the solutions of the problems above.

100 Functions of several variables, domain. Problem solving 3.

Problem 7: Specify the domain of the function $f(x,y) = \ln(1+xy)$.

Extra material: notes with the solution of the problem above.

Problem 8: Specify the domain of the function $f(x,y) = \ln\frac{x+y}{x-y}$.

101 Functions of several variables, graphs.

Extra material: Graphs to $f(x,y) = x^2+y^2$ and $f(x,y) = \sqrt{x^2+y^2}$, notes to the video above.

102 Plotting functions of two variables. Problem solving part 1.

Problem 1: Sketch the graph of $f(x,y) = 5$.

Problem 2: Sketch the graph of $f(x,y) = x$.

Problem 3: Sketch the graph of $f(x,y) = y^2$.

Problem 4: Sketch the graph of $f(x,y) = x^2+y^2+5$.

Problem 5: Sketch the graph of $f(x,y) = -x^2-y^2+1$.

103 Plotting functions of two variables. Problem solving part 2.

Problem 6: Sketch the graph of $f(x,y) = x^2+y^2-4x+4y+10$.

Problem 7: Sketch the graph of $f(x,y) = \frac{1}{1+x^2+y^2}$.

Extra material: notes with solved problems 1–7.

104 Functions of several variables, level curves.

Problem 1: Describe some level curves for $f(x,y) = \sqrt{x^2+y^2}$.

Problem 2: Describe some level curves for $f(x,y) = x^2+y^2$.

Problem 3: Describe some level curves for $f(x,y) = \sqrt{1-x^2-y^2}$.

Problem 4: Describe some level curves for $f(x,y) = \ln(x^2+y^2)$.

105 Level curves; problem solving 1.

Problem 1: Describe some level curves for $f(x,y) = x-y$.

106 Level curves; problem solving 2.

Problem 2: Describe some level curves for $f(x,y) = x^2+4y^2$.

107 Level curves; problem solving 3.

Problem 3: Describe some level curves for $f(x,y) = xy$.

108 Level curves; problem solving 4.

Problem 4: Describe some level curves for $f(x,y) = \dfrac{x-y}{x+y}$.

109 Level curves; problem solving 5.

Problem 5: Describe some level curves for $f(x,y) = \dfrac{y}{x^2+y^2}$.

Extra material: notes with solved problems 1–5.

110 Level surfaces, definition and problem solving.

Problem 1: Describe some level surfaces for $f(x,y,z) = x + 2y + 3z$.

Problem 2: Describe some level surfaces for $f(x,y,z) = x^2 + y^2 + z^2$.

Problem 3: Describe some level surfaces for $f(x,y,z) = x^2 + y^2$.

Problem 4: Describe some level surfaces for $f(x,y,z) = \dfrac{x^2+y^2}{z^2}$.

Extra material: notes with solved problems 1–4.

S13 Limit, continuity

You will learn: calculate limit values, determine if a function has limit value or is continuous at one point, use common sum-, product-, ... rules for limits.

111 Functions of several variables, limit and continuity, part 1.

112 Functions of several variables, limit and continuity, part 2.

113 Functions of several variables, limit and continuity, part 3.

Problem 1: Show that the limit of $f(x,y) = \dfrac{x}{x^2+y^2}$ in the origin does not exist.

Problem 2: Show that the limit of $f(x,y) = \dfrac{2xy}{x^2+y^2}$ in the origin does not exist.

Extra material: notes with the solutions of the problems above.

114 Limits, problem solving part 1: Problems 1–4.

Problem 1: Evaluate the limit of $f(x,y) = \dfrac{2x-y+5}{1+x^2+y^2}$ in the origin.

Problem 2: Show that the limit of $f(x,y) = \dfrac{2xy}{x^2+y^2}$ in the origin does not exist.

Problem 3: Show that the limit of $f(x,y) = \dfrac{x^4y^2}{(x^4+y^2)^2}$ in the origin does not exist.

Problem 4: Show that the limit of $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ in the origin does not exist.

Extra material: notes with the solutions of problems 1 and 4 above.

115 Limits, problem solving part 2: Problem 0, Problems 5–7.

Problem 0: Show that the limit of $f(x,y) = \dfrac{x^3}{x^2+y^2}$ in the origin is equal to zero.

Problem 5: Show that the limit of $f(x,y) = \dfrac{xy}{\sqrt{x^2+3y^2}}$ in the origin is equal to zero.

Problem 6: Show that the limit of $f(x,y) = (x^2+y^2)\ln(x^2+y^2)$ in the origin is equal to zero.

Problem 7: Show that the limit of $f(x,y) = \dfrac{\ln(1+xy)}{xy+x^3y^3}$ in the origin is equal to one.

Extra material: notes with the solution of Problem 0 above.

116 Limits, problem solving part 3.

Problem 1: How can the function $f(x,y) = \dfrac{x^2+y^2-x^3y^3}{x^2+y^2}$, $(x,y) \neq (0,0)$ be defined in the origin so that the expanded function is continuous on the whole plane?

Problem 2: Show that the limit of $f(x,y) = \dfrac{\sin(xy)}{x^2+y^2}$ in the origin does not exist.

Extra material: notes with the solutions of the problems above.

117 Limits, problem solving part 4.

Problem: What is the domain of $f(x,y) = \dfrac{x-y}{x^2-y^2}$? Does $f(x,y)$ have a limit as $(x,y) \to (1,1)$? Can the domain of the function be extended so that the resulting function is continuous at $(1,1)$? Can the domain of the function be extended so that the resulting function is continuous on the whole plane?

Extra material: notes with the solution of the problem above.

S14 Partial derivative, tangent plane, normal line, gradient, Jacobian

You will learn: calculate first-order partial derivatives, compute scalar products (two formulas) and cross product, give formulas for normals and tangent planes; understand functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, gradients and Jacobians.

118 Partial derivatives, introduction 1: the definition and notation.

119 Partial derivatives, introduction 2: arithmetical consequences.

Problem 1: Find all the first-order partial derivatives of the function $f(x,y,z) = x^3y^4z^5$.

Problem 2: Find all the first-order partial derivatives of the function $f(x,y) = e^{x+y}\sin(xy)$.

Problem 3: Find all the first-order partial derivatives of the function $f(x,y) = \dfrac{x}{x+y}$.

Extra material: notes with solved Problem 1.

120 Partial derivatives, introduction 3: geometrical consequences (tangent plane).

121 Partial derivatives: not good enough. Introduction 4.

Existence of partial derivatives does not guarantee continuity. An example of a function which has both partial derivatives in origin but is not continuous in the origin: $f(x,y) = \dfrac{2xy}{x^2+y^2}, (x,y) \neq (0,0), f(0,0) = 0.$

122 Partial derivatives, introduction 5: a pretty terrible example.

Problem 1: Find all the first-order partial derivatives of the function $w(x,y,z) = x^{y \ln z}$ and evaluate them in the point $(x,y,z) = (e,2,e)$.

Extra material: notes with the solution of the problem above.

123 Tangent plane, part 1.

124 Normal vector.

Extra material: notes to the video above.

125 Tangent plane, part 2; normal equation.

126 Normal line.

Problem: Find the normal line to the paraboloid $f(x,y) = x^2+y^2$ in the point $(1,2,5)$.

127 Tangent planes: problem solving. Problem 1.

Problem 1: Find equations of the tangent plane and the normal line to the graph of function f at the point (a,b): $f(x,y) = x^2-y^2$, $(a,b) = (-2,\ 1)$.

128 Tangent planes: problem solving. Problem 2.

Problem 2: Find equations of the tangent plane and the normal line to the graph of function f at the point (a,b): $f(x,y) = \cos\frac{x}{y}$, $(a,b) = (\pi,\ 4)$.

129 Tangent planes: problem solving. Problem 3.

Problem 3: Find equations of the tangent plane and the normal line to the graph of function f at the point

(a, b): $f(x, y) = \dfrac{x}{x^2 + y^2}$, $(a, b) = (1, 2)$.

130 Tangent planes: problem solving. Problem 4.

Problem 4: Find equations of the tangent plane and the normal line to the graph of function f at the point (a, b): $f(x, y) = \ln(x^2 + y^2)$, $(a, b) = (1, -2)$.

131 Tangent planes: problem solving. Problem 5.

Problem 5: Find equations of the tangent plane and the normal line to the graph of function f at the point (a, b): $f(x, y) = \ln(x^2 y) + (2x - y)^4 - 2y$, $(a, b) = (1, 1)$.

Extra material: notes with the solutions of the problems 1–5 above.

132 The gradient.

133 A way of thinking about functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Example 1: A linear map $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T_A(x, y, z) = (3x - 5y + z, \ 4x + 6y + 7z, \ 3x - z, \ 8y - 9z)$.

Example 2: A nonlinear function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\mathbf{f}(x, y, z) = (x^2 + yz, \ y^2 - x \ln z)$.

Example 3: Nonlinear function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^4$ defined by
$\mathbf{f}(x, y, z) = (3x - 5 \sin y + z^4, \ 4xy + 6x^3 y^2 - xyz, \ e^x y - \cos(x^2 y), \ 8x + e^{xyz})$.

134 The Jacobian.

Four examples of Jacobian (for the functions mentioned in the video above, and for the change of variables to polar coordinates).

S15 Higher partial derivatives

You will learn: compute higher order partial derivatives, use Schwarz' theorem. Solve and verify some simple PDE's.

135 Higher-order partial derivatives, intro.

Compute all second-order partial derivatives for $f(x, y, z) = x^3 y^4 z^5$.

Extra material: notes with the solutions of the problem above.

136 Higher-order partial derivatives, definition and notation.

137 Higher-order partial derivatives, mixed partials, Hessian matrix.

138 The difference between Jacobian matrices and Hessian matrices.

139 Equality of mixed partials; Schwarz' theorem.

Show that $f''_{31} = f''_{13}$ for $f(x, y, z) = 3x^2 y^3 z^2 + y^5 z$.

Extra material: notes with the solutions of the problem above.

140 Schwarz' theorem: Peano's example.

141 Schwarz' theorem: the proof.

142 PDE (Partial Differential Equations), intro.

143 PDE (Partial Differential Equations), basic ideas.

144 PDE (Partial Differential Equations), problem solving.

Problem 1: Is it possible to find a function of two variables, which satisfies two conditions: $f'_x = 3x^2 y + y^2$ and $f'_y = x^3 + 2xy + 3y^2$?

Problem 2: Is it possible to find a function of two variables, which satisfies two conditions: $f'_x = x + x^2 y$ and $f'_y = \frac{1}{3} x^3 + xy$?

Extra material: notes with solved problems 1 and 2.

145 Laplace equation and harmonic functions 1.

Problem 1: Show that the following function is harmonic: $z = 3x^2 y - y^3$.

Problem 2: Show that the following function is not harmonic: $z = 4x^2y^2 - yx + 3y^2$.

Extra material: notes with solved problems 1 and 2.

146 Laplace equation and harmonic functions 2.

Problem 1: Show that the following function is harmonic: $z = e^{kx}\cos(ky)$.

Problem 2: Show that the following function is harmonic: $z = e^{kx}\sin(ky)$.

Extra material: notes with solved problems 1 and 2.

147 Laplace equation and Cauchy—Riemann equations.

Problem: Let the functions $u(x, y)$ and $v(x, y)$ have continuous second partial derivatives and satisfy the Cauchy—Riemann equations. Show that both functions are harmonic.

Extra material: notes with solved problem above.

148 Dirichlet problem.

Problem: Let $D = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$. Show that $u(x, y) = \ln(x^2 + y^2)$ is the solution to the Dirichlet problem on D with boundary conditions $u(x, y) = 0$ on $x^2 + y^2 = 1$ and $u(x, y) = 2\ln(2)$ on $x^2 + y^2 = 4$.

S16 Chain rule: different variants

You will learn: calculate the chain rule using dependency diagrams and matrix multiplication.

149 The Chain Rule: a general introduction.

150 The Chain Rule: variants 1 and 2.

Variant 1: composition $\vec{v} \circ s : \mathbb{R} \xrightarrow{s} \mathbb{R} \xrightarrow{\vec{v}} \mathbb{R}^n$, derivative $\frac{d}{dt}(\vec{v}(s(t))) = s'(t) \cdot \vec{v}\,'(s(t))$.

Variant 2: composition $z = s \circ f : \mathbb{R}^2 \xrightarrow{f} \mathbb{R} \xrightarrow{s} \mathbb{R}$, derivatives $\dfrac{\partial z}{\partial x} = \frac{ds}{dt} \cdot \dfrac{\partial f}{\partial x}$, $\dfrac{\partial z}{\partial y} = \frac{ds}{dt} \cdot \dfrac{\partial f}{\partial y}$.

Example on Variant 2: Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial x}$ for $z = \arctan \dfrac{y}{x}$.

151 The Chain Rule: variant 3.

Variant 3: composition $f \circ \vec{v} : \mathbb{R} \xrightarrow{\vec{v}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$, derivative $\frac{d}{dt}\left[f(\vec{v}(t)\right] = \nabla f(\vec{v}(t)) \cdot \vec{v}'(t)$.

Example on Variant 3: Determine $\dfrac{d}{dt} f(\vec{v}(t))$ for $f(x, y, z) = xz + \cos y$ and $\vec{v}(t) = (\sin t,\ t^2,\ \ln(t^2 + 1))$.

152 The Chain Rule: variant 3 (proof).

153 The Chain Rule: variant 4.

Variant 4: composition $f \circ \Phi : \mathbb{R}^n \xrightarrow{\Phi} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$, derivative $\nabla(f \circ \Phi) = \nabla f \cdot D\Phi$.

154 The Chain Rule: an example with a diagram.

Compute $\dfrac{\partial T}{\partial t}$ om $T = T(x, y, z) = T(x(u, v), y(t), z(w, t)) = T(x(u(s, t), v), y(t), z(w, t))$.

155 The Chain Rule: problem solving.

Let $f(x, y)$ be a continuous function with continuous partial derivatives of first order. Suppose that for all real numbers $t > 0$ holds $f(tx, ty) = t^3 f(x, y)$. Show that it implies $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 3f(x, y)$.

156 The Chain Rule: problem solving 1.

Problem 1: Compute $\dfrac{\partial}{\partial x}(f(xy^2,\ x^3))$ where f is a partially differentiable function.

Extra material: notes with solved Problem 1.

157 The Chain Rule: problem solving 2.

Problem 2: Function $f(u, v)$ is a two variable function differentiable in the whole plane. Let $h(x, y, z) =$

$f\left(\frac{x}{y}, \frac{y}{z}\right)$, $y > 0,\ z > 0$. Show using the chain rule for the composite function h that $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} + z\frac{\partial h}{\partial z} = 0$.

Extra material: notes with solved Problem 2.

158 The Chain Rule: problem solving 3.

Problem 3: If $z = f(x, y),\ \ x = 2s + 3t,\ \ y = 3s - 2t$, determine $\frac{\partial^2 z}{\partial s^2},\ \frac{\partial^2 z}{\partial t^2},\ \frac{\partial^2 z}{\partial s \partial t}$.

Extra material: notes with solved Problem 3.

159 The Chain Rule: problem solving 4.

Problem 4: Solve the PDE $\frac{\partial f}{\partial x} - 3\frac{\partial f}{\partial y} = x$ with condition $f(0, y) = e^y$ by using the following change of variables: $u = 3x + y,\ \ v = x$.

Extra material: notes with solved Problem 4.

160 The Chain Rule: problem solving 6.

Problem 6: Solve the PDE $y\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial x} = 0\ \ (x > 0,\ y > 0)$ by using the following change of variables: $u = xy,\ \ v = x$.

Extra material: notes with solved Problem 6.

161 The Chain Rule: problem solving 7.

Problem 7: Compute $\frac{\partial z}{\partial u}$ where $z = g(x, y),\ \ y = f(x),\ \ x = h(u, v)$. (All the involved functions are continuously differentiable.)

Extra material: notes with solved Problem 7.

162 The Chain Rule: problem solving 5.

Problem 5: Solve the PDE $\frac{\partial^2 f}{\partial x^2} - 4x\frac{\partial^2 f}{\partial x \partial y} + 4x^2\frac{\partial^2 f}{\partial y^2} - 2\frac{\partial f}{\partial y} = 0$ by using the following change of variables: $u = x^2 + y,\ \ v = x$.

Extra material: notes with solved Problem 5.

163 The Chain Rule: problem solving 8.

Problem 8: Compute $\frac{\partial z}{\partial x}$ (in two different ways) knowing that $z = \arctan\frac{u}{v},\ u = 2x + y,\ v = 3x - y$.

Extra material: notes with solved Problem 8.

S17 Linear approximation, linearisation, differentiability, differential

You will learn: determine if a function is differentiable in a point, linearisation of a real-valued function, use linearisation to derive an approximate value of a function, use the test for differentiability (continuous partial derivatives), and properties of differentiable functions.

164 Linearization and differentiability in Calc1.

165 Differentiability in Calc3: intro.

166 Differentiability in two variables, an example.

Use the definition of differentiability to show that $f(x, y) = x^2 + y^2$ is differentiable in every point $(x, y) \in \mathbb{R}^2$.

167 Differentiability in Calc3 implies continuity.

168 Partial differentiability does NOT imply differentiability.

Existence of partial derivatives does not guarantee continuity. An example of a function which has both partial derivatives in origin but is not continuous in the origin: $f(x, y) = \frac{2xy}{x^2 + y^2}, (x, y) \neq (0, 0), f(0, 0) = 0$.

169 An example: continuous, not differentiable.

$f(x, y) = \sqrt{x^2 + y^2}$ is continuous but not differentiable in the origin.

170 Differentiability in several variables, a test.

171 A wrap-up: Differentiability, partial differentiability and continuity in Calc3.

172 Differentiability in two variables, a geometric interpretation.

173 Linearization, two examples.

Problem 1: Determine the linearization of $f(x, y) = \sqrt{2x^2 + e^{2y}}$ in $(x, y) = (2.2, -0.2)$.

Problem 2: Let $f(x, y) = \ln|x^2 + xy|$. Approximate $f(2.01, 1.03) - f(2, 1)$.

174 Linearization: problem solving 1.

Problem 1: Use suitable linearization to approximate the value for the functions f at the point indicated: $f(x, y) = x^2y^3, \;\; (x, y) = (3.1, \; 0.9)$.

Extra material: notes with solved Problem 1.

175 Linearization: problem solving 2.

Problem 2: Use suitable linearization to approximate the value for the functions f at the point indicated: $f(x, y) = \arctan \frac{y}{x}, \;\; (x, y) = (3.01, \; 2.99)$.

Extra material: notes with solved Problem 2.

176 Linearization: problem solving 3.

Problem 3: Use suitable linearization to approximate the value for the functions f at the point indicated: $f(x, y) = \sin(\pi xy + \ln y), \;\; (x, y) = (0.01, \; 1.05)$.

Extra material: notes with solved Problem 3.

177 Linearization by Jacobian matrix: problem solving.

Problem: Let $f(x, y, z) = (x^2 + yz, \;\; y^2 - x\ln z)$. Compute the Jacobian matrix of this function and use it to determine the approximative value of $f(1.98, \; 2.01, \; 1.03)$.

Extra material: notes with solved problem above.

178 Differentials: problem solving 1.

Problem 1: The edges of a rectangular box are each measured to within an accuracy of 1% of their values. What is the approximate maximum percentage error in: a) the calculated volume of the box, b) the calculated area of one of the faces of the box, c) the calculated length of a diagonal of the box?

179 Differentials: problem solving 2.

Problem 2: By approximately what percentage will the value of $w = \frac{x^2y^3}{z^4}$ increase or decrease if x increases by 1%, y increases by 2% and z increases by 3%? (Assume that $x, \; y, \; z > 0$.)

Extra material: notes with solved Problem 2.

S18 Gradient, directional derivatives

You will learn: calculate the gradient, find the directional derivative in a certain direction, properties of gradient, understand the geometric interpretation of the directional derivative, give a formula for the tangent and normal lines to a level curve.

180 Gradient.

181 The gradient in each point is orthogonal to the level curve through this point.

182 The gradient in each point is orthogonal to the level surface through this point.

183 Tangent plane to the level surface, an example.

Problem: Determine a normal equation for the tangent plane to the level surface for $f(x, y, z) = x^2 + y^2 + z^2$ at the point $(1, 3, 4)$.

184 Directional derivatives, intro.

185 Directional derivatives, the direction.

186 How to normalize a vector and why it works.

Extra material: notes to the video above.

187 Directional derivatives, the definition.

188 Partial derivatives as special case of directional derivatives.

189 Directional derivatives, an example.

Problem: Compute the directional derivative of the function $f(x,y) = x^2y$ at the point $(x,y) = (2,2)$ in the direction $\vec{v} = \frac{1}{\sqrt{2}}(1,\ 1)$ (from the definition).

Extra material: notes with solved problem above.

190 Directional derivatives: an important theorem for computations and interpretations.

191 Directional derivatives: an earlier example revisited.

Problem: Compute the directional derivative of the function $f(x,y) = x^2y$ at the point $(x,y) = (2,2)$ in the direction $\vec{v} = \frac{1}{\sqrt{2}}(1,\ 1)$ (using the theorem from the previous video).

Extra material: notes with solved problem above.

192 Geometrical consequences of the theorem about directional derivatives.

193 Geometrical consequences of the theorem about directional derivatives, an example.

Problem: The temperature in the room is described by function $f(x,y,z) = \dfrac{z^2}{x^2+y^2}$. Consider the point in the room with coordinates $(1,1,1)$. What is the change in temperature in this point in direction $(1,1,1)$? In what direction in this point is the change in temperature the fastest and how fast is the temperature changing in this point in this direction?

194 Directional derivatives: an example.

Problem: Compute the rate of change of $f(x,y) = y^4 + 2xy^3 + x^2y^2$ at $(x,\ y) = (0,\ 1)$ measured in each of the following directions: a) $\vec{v} = (1,\ 2)$, b) $\vec{v} = (1,\ -2)$, c) $\vec{v} = (3,\ 0)$ and d) $\vec{v} = (1,\ 1)$.

Extra material: notes with solved problem above.

195 Normal line and tangent line to a level curve: how to get their equations.

196 Normal line and tangent line to a level curve: how to get their equations: an example.

Problem: Let

$$f(x,y) = \frac{3}{1+x^2+y^2}.$$

a) In what direction and how fast does $f(x,y)$ increase at the highest rate at point $(1,1)$?

b) Compute the directional derivative of f in $(1,1)$ in the direction defined by $\mathbf{i} + \sqrt{3}\mathbf{j}$.

c) Determine equations of the tangent line and the normal line to the level curve through the point $(1,1)$.

d) Determine an equation of the tangent plane to the surface $z = f(x,y)$ in point $(1,1,1)$.

Extra material: notes to the video above.

197 Gradient and directional derivatives, problem solving, Problem 1.

Problem 1: Let $f(x,y) = x^2 - y^2,\ \ (a,b) = (2,-1)$. Compute:

a) the gradient of the function f at the point (a,b),

b) an equation of the tangent plane to the graph surface in the point $(a,b,f(a,b))$,

c) an equation of the line tangent to the level curve through (a,b).

Extra material: notes with solved Problem 1.

198 Gradient and directional derivatives, problem solving, Problem 2.

Problem 2: Let $f(x,y) = \ln(x^2+y^2),\ \ (a,b) = (1,-2)$. Compute:

a) the gradient of the function f at the point (a,b),

b) an equation of the tangent plane to the graph surface in the point $(a,b,f(a,b))$,

c) an equation of the line tangent to the level curve through (a, b).

Extra material: notes with solved Problem 2.

199 Gradient and directional derivatives, problem solving, Problem 3.

Problem 3: Find an equation of the tangent plane to the level surface of $f(x, y, z) = x^2y + y^2z + z^2x$ at $(1, -1, 1)$.

Extra material: notes with solved Problem 3.

200 Gradient and directional derivatives, problem solving, Problem 4.

Problem 4: Find the rate of change of $f(x, y) = x^2y$ at $(-1, -1)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j}$.

Extra material: notes with solved Problem 4.

201 Gradient and directional derivatives, problem solving, Problem 5.

Problem 5: In what directions at the point $(2, 0)$ does the function $f(x, y) = xy$ have rate of change -1? Are there directions in which the rate is -3? And -2?

Extra material: notes with solved Problem 5.

202 Gradient and directional derivatives, problem solving, Problem 6.

Problem 6: Find the gradient in the point (a, b) for the differentiable function $f(x, y)$ knowing that the directional derivative (in this point) in direction of the vector $(\mathbf{i}+\mathbf{j})/\sqrt{2}$ is equal to $3\sqrt{2}$, and the directional derivative (in this point) in direction of the vector $(3\mathbf{i} - 4\mathbf{j})/5$ is equal to 5.

Extra material: notes with solved Problem 6.

203 Gradient and directional derivatives, problem solving, Problem 7.

Problem 7: Find a vector tangent to the curve of intersection of the two cylinders $x^2+y^2 = 2$ and $y^2+z^2 = 2$ at the point $(1, -1, 1)$.

Extra material: notes with solved Problem 7.

S19 Implicit functions

You will learn: calculate the Jacobian determinant, compute partial derivatives with dependent and independent variables of implicit functions.

204 What is the Implicit Function Theorem?

205 Jacobian determinant.

206 Jacobian determinant for change to polar and to cylindrical coordinates.

Compute the Jacobian determinant for change of variables to polar coordinates $x = r\cos\theta,\ y = r\sin\theta$.

The same for cylindrical coordinates $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.

207 Jacobian determinant for change to spherical coordinates.

Compute the Jacobian determinant for change of variables to spherical coordinates $x = R\sin\phi\cos\theta$, $y = R\sin\phi\sin\theta,\ z = R\cos\phi$.

208 Jacobian determinant and change of area.

209 The Implicit Function Theorem variant 1. (Curve as a level curve $F(x, y) = 0$; solve for y as function of x around some point; compute $y'(x)$ in this point.)

210 The Implicit Function Theorem variant 1, an example.

Compute $\dfrac{dx}{dy}$ if $xy^3 + x^4y = 2$.

Extra material: notes with the solved problem.

211 The Implicit Function Theorem variant 2. (Surface as a level surface $F(x, y, z) = 0$; solve for z as function of x and y around some point; compute the partials z'_x and z'_y in this point.)

212 The Implicit Function Theorem variant 2, example 1.

Show that the equation $x^3 - xyz^2 + z^3 + 3 = 0$ in some neighborhood of the point $(1, -1, -2)$ defines z as a C^∞-function (differentiable infinitely many times) $z = z(x, y)$ of variables x and y, such that $z(1, -1) = -2$. Compute $\nabla z(1, -1)$.

213 The Implicit Function Theorem variant 2, example 2.

Compute $\dfrac{\partial z}{\partial y}$ if $z^2 + xy^3 = \dfrac{xz}{y}$.

Extra material: notes with the solved problem.

214 The Implicit Function Theorem variant 3. (Curve as an intersection of two level surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$; parameterize this curve with x, i.e. $\vec{r}(x) = (x, y(x), z(x))$, around some point; compute $y'(x)$ and $z'(x)$ in this point.)

215 The Implicit Function Theorem variant 3, example.

Consider the following system of equations

$$\begin{cases} \sin(x+y) + \sin(y+z) + z = 0 \\ \cos(x+y) + \cos(y+z) + y - 2 = 0 \end{cases}$$

with two equations and three variables. The point $(x, y, z) = (0, 0, 0)$ satisfies the system. Can x and y be solved explicitly as functions of z near this point? If the answer is yes, compute $x'(z)$ and $y'(z)$ in the point $(0, 0, 0)$.

216 The Implicit Function Theorem variant 4. (Change of variables in $\mathbb{R}^2$.)

217 The Inverse Function Theorem.

218 The Implicit Function Theorem, summary.

219 Notation in some unclear cases.

Compute $\dfrac{\partial u}{\partial x}$ if $G(x, y, z, u, v) = 0$.

220 The Implicit Function Theorem, problem solving 1.

If $x = u^3 + v^3$ and $y = uv - v^2$ are solved for u and v in terms of x and y, evaluate

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial(u, v)}{\partial(x, y)}$$

in the point where $u = 1$ and $v = 1$.

Extra material: notes with the solved problem.

221 The Implicit Function Theorem, problem solving 2.

Find $x'(y)$ from the system $F(x, y, z, w) = 0, \;\; G(x, y, z, w) = 0, \;\; H(x, y, z, w) = 0$.

Extra material: notes with the solved problem.

222 The Implicit Function Theorem, problem solving 3.

Show that the equation $x^2 + y^2 + 2z^3 - 15 = 0$ defines an explicit C^1-function $z = z(x, y)$ in some neighborhood of the point $(x, y, z) = (3, 2, 1)$ and compute z'_x and z'_y in this point.

Extra material: notes with the solved problem.

223 The Implicit Function Theorem, problem solving 4.

Consider the following system of equations

$$\begin{cases} x + y^2 + z^3 - 6 = 0 \\ x^2 + y + z^3 - 4 = 0 \end{cases}$$

with two equations and three variables. The point $(x, y, z) = (1, 2, 1)$ satisfies the system. Can y and z be solved explicitly as functions of x near this point? If the answer is yes, compute $y'(x)$ and $z'(x)$ in the point $(1, 2, 1)$.

Extra material: notes with the solved problem.

S20 Taylor's formula, Taylor's polynomial, quadratic forms

You will learn: derive Taylor's polynomials and Taylor's formula, understand quadratic forms and learn how to determine if they are positive definite, negative definite or indefinite.

224 Taylor's formula, intro.

225 Quadratic forms and Taylor's polynomial of second degree.

226 Taylor's polynomial of second degree, theory.

227 Taylor's polynomial of second degree, example 1.

Problem 1: Determine Taylor's polynomial of second degree for $f(x,y) = e^{xy} + x^2 + 2xy^3 + 3y$ around the point $(2,0)$.

228 Taylor's polynomial of second degree, example 2.

Problem 2: Determine Taylor's polynomial of second degree for $f(x,y) = xy - x^2y - y^2$ around the points $(0,0)$ and $(1,0)$.

229 Taylor's polynomial of second degree, example 3.

Problem 3: Determine Taylor's polynomial of second degree for $f(x,y) = 7 + e^{x^2+y^2}$ around the point $(0,0)$.

Extra material: notes with solved problem above.

230 Classification of quadratic forms (positive definite etc).

231 Classification of quadratic forms, problem solving 1.

Problem 1: Classify the quadratic form defined by $\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$.

232 Classification of quadratic forms, problem solving 2.

Problem 2: Classify the quadratic form defined by $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 3: Classify the quadratic form defined by $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

233 Classification of quadratic forms, problem solving 3.

Problem 4: Classify the quadratic form defined by $\begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$.

Problem 5: Classify the quadratic form defined by $\begin{bmatrix} 6 & 3 \\ 3 & 1 \end{bmatrix}$.

Extra material: notes with solved problems 1–5.

C3 Optimization of functions of several variables

(Chapter 13: 13.1–3)

You will learn: classify critical points: local max and min, saddle points; find max and min values for a given function and region; use Lagrange multipliers with one or more conditions.

S21 Optimization on open domains (critical points)

234 Extreme values of functions of several variables.

235 Extreme values of functions of two variables, without computations.

Example 1: Find max and min for $f(x,y) = \sqrt{1 - x^2 - y^2}$.

Example 2: Find max and min for $f(x,y) = \dfrac{1}{x^2 + y^2 + 1}$.

Example 3: Find max and min for $f(x,y) = x^2 + y^2 - 6x + 4y + 10$.

Example 4: Find max and min for $f(x,y) = \sqrt{x^2 + y^2}$.

Example 5: Find max and min for $f(x,y) = x + y$.

236 Critical points and their classification (max, min, saddle).

237 Second derivative test for C^3 functions of several variables.

238 Second derivative test for C^3 functions of two variables.

239 Critical points and their classification: some simple examples.

Example 1: Classify the CP $(0,0)$ for $f(x,y) = x^2 + y^2 + 1$.

Example 2: Classify the CP $(0,0)$ for $f(x,y) = 3 - x^2 - y^2$.

Example 3: Classify the CP $(0,0)$ for $f(x,y) = x^2 - y^2 + 2$.

240 Critical points and their classification: more examples 1.

Problem 1: Classify the CPs $(0,0)$ and $\left(\frac{1}{12}, -\frac{1}{6}\right)$ to $f(x,y) = 3x^2 + 3xy + y^2 + y^3$.

241 Critical points and their classification: more examples 2.

Problem 2: Find and classify all the CPs to $f(x,y) = 3xy - x^2 - 3y^2 + x - 12$.

242 Critical points and their classification: more examples 3.

Problem 3: Find and classify all the CP to $f(x,y) = xy - x^2y - y^2$.

243 Critical points and their classification: a more difficult example (4).

Problem 4: Find and classify all the CP to $f(x,y) = x^2ye^{-(x^2+y^2)}$.

Extra material: notes with solved problem 4.

S22 Optimization on compact domains

244 Extreme values for continuous functions on compact domains.

245 Eliminate a variable on the boundary.

Example: Find max and min for $f(x,y) = xy^2 - x - y$ on the square D: $0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant 2$.

246 Parameterize the boundary.

Example: Find max and min for $f(x,y) = x + y$ on the disk $x^2 + y^2 \leqslant 1$.

S23 Lagrange multipliers (optimization with constraints)

247 Lagrange multipliers 1. (Maximize and minimize $f(x,y)$ subject to $g(x,y) = 0$.)

248 Lagrange multipliers 1, an old example revisited.

Problem: Find max and min for $f(x,y) = x + y$ on the disk $x^2 + y^2 \leqslant 1$.

Extra material: notes with solved problem above.

249 Lagrange multipliers 1, another example.

Problem: Find max and min for $f(x,y) = x^2y$ on the circle $x^2 + y^2 = 3$.

Extra material: notes with solved problem above.

250 Lagrange multipliers 2. (Maximize and minimize $f(x,y,z)$ subject to $g(x,y,z) = 0$.)

Problem: Determine max and min for $f(x,y,z) = x + 2y + 3z$ with constraint $x^2 + y^2 + z^2 = 14$, so on the sphere with centre in the origin and radius $\sqrt{14}$.

251 Lagrange multipliers 2, an example.

Problem: Find max and min for $f(x,y,z) = xyz$ on the ball $x^2 + y^2 + z^2 \leqslant 12$.

Extra material: notes with solved problem above.

252 Lagrange multipliers 3. (Maximize and minimize $f(x,y,z)$ subject to $g(x,y,z) = 0$ and $h(x,y,z) = 0$.)

253 Lagrange multipliers 3, an example.

Problem: Find max and min for $f(x,y,z) = y$ under constraints $g(x,y,z) = x+y+z-1 = 0$ and $h(x,y,z) = x^2+y^2+z^2-1 = 0$.

254 Summary: optimization.

S24 Wrap-up Multivariable calculus / Calculus 3, Part 1 of 2

255 Final words.

Extra material to C3: a document with solutions to the following problems:

* Let $f(x,y) = 3x^3 + 3x^2y - y^3 - 15x$.
  a) Find and classify the critical points of f. Use any method taught during the course (the second-derivative test or completing the square).
  b) One of the critical points is $(a,b) = (1,1)$. Write down the second-degree Taylor approximation of f about this point and motivate, both with computations and with words, how one can see from this approximation what kind of critical point $(1,1)$ is. Use completing the square.

* Let $f(x,y) = 3x^2 + 3xy + y^2 + y^3$. Find and classify all critical points of f. Use any method taught during the course (the second-derivative test or completing the square).

* Let
$$f(x,y) = 2e^{2y} - 4e^x e^y + e^{4x}.$$
  a) Find and classify the critical points of f. Use any method taught during the course (the second-derivative test or completing the square).
  b) One of the critical points is $(a,b) = (0,0)$. Write down the second-degree Taylor approximation of f about this point and motivate, both with computations and with words, how one can see from this approximation what kind of critical point $(0,0)$ is. Use completing the square.

* Determine and classify all the critical points for
$$f(x,y,z) = x^3 + 3x^2 + 4y^2 + 6z^2 - 6xy - 6xz + 8yz + 4z.$$

* Find the size of a rectangular box with no top (i.e., one of the six faces is missing) having the least possible total surface area, knowing that the volume of the box is 32.

* Let
$$D = \{(x,y) \mid x \geqslant 0, \quad y \geqslant 0, \quad x^2 + 4y^2 \leqslant 1\}.$$
  a) Sketch D. Explain briefly how we can see that D is closed and bounded.
  b) Find the largest and the smallest values of $f(x,y) = x^2 + y$ on D.

* Maximize and minimize $f(x,y) = (2x+3y+1)^2$ on the circle $x^2 + y^2 = 1$.